1. *Let S be a non-empty finite set and let $g : S \to S$ be a function. Show that there exists $x \in S$ and $n \in \mathbb{N}$ such that*
$$g^n(x) = x$$
where $g^n = g \circ g \circ ... \circ g$ (n times).

   Solution: Let $x \in S$. Let
$$A = \{g^n(x) | n \in \mathbb{N}\}$$
Since $A \subseteq S$, A is finite. So there exists $n, m \in \mathbb{N}$ such that $n < m$ and $g^n(x) = g^m(x)$ i.e, $g^n(x) = g^{m-n} \circ g^n(x)$. Now taking $y = g^n(x)$, we have $g^{m-n}(y) = y$. □

2. *Prove or disprove the claim that $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by*
$$f(m, n) = 2^{m-1}(2n - 1)$$
is a bijection.

   Solution: Let us assume that $f(m, n) = f(p, q)$ for some $m, n, p, q \in \mathbb{N}$. Then,
$$2^{m-1}(2n - 1) = 2^{p-1}(2q - 1)$$
Dividing both sides by 2^{m-1}
$$2n - 1 = \frac{2^{p-1}}{2^{m-1}} \times (2q - 1) \tag{1}$$
Since L.H.S is odd so is R.H.S. But that will happen only if m=p. Putting this in (1) we get $n = q$. Therefore $(m, n) = (p, q)$. Hence f is one to one.

   Let $k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that 2^{n-1} divides k and 2^n does not divide k. Then $k/2^{n-1}$ is odd. So there exists a $m \in \mathbb{N}$ such that $k/2^{n-1} = 2m-1$. Therefore $k = (2m-1) \times 2^{n-1}$. Hence $f(n, m) = k$.

   So f is onto. □

3. *Show that given any two numbers $x, y \in \mathbb{R}$ with $x < y$ there exists a rational number r such that $x < r < y$.*

   Solution: Let us first suppose that $0 \leq x$. By the Axiom of Archimedes there is an integer $q > 1/(y - x)$. Then $(1/q) < y - x$.The set of integers n such that $y \leq (n/q)$is a non empty (by Axiom of Archimedes) set of positive integers, and so has a least element p. Then $(p - 1)/q < y \leq (p/q), andx = y - (y - x) < (p/q) - (1/q) = (p - 1)/q$. Thus $r = (p - 1)/q$ lies between x and y. If $x < 0$, we can find an integer n such that $n > -x$. Then $n + x > 0$, and there is rational r with $n + x < r < n + y$, and $r - n$ is a rational between x and y. □

4. *Let $\{b_n\}$ be a sequence of non-zero real numbers converging to a non-zero real number c. Show that the sequence $\{b_n\}$ converges to $1/c$.*

   Solution: Let $\varepsilon > 0$ be arbitary. Then there exist $N \in \mathbb{N}$ such that $\mid \ b_n \ - c \mid < \varepsilon$ for every $n > N$. Since $\{b_n\} \longrightarrow c \neq 0$, there exists $M \in \mathbb{N}$ such that $\mid b_n \mid > r > 0$ for every $n > M$,for some $r > 0$. Therefore
   $$\left| \frac{1}{b_n} - \frac{1}{c} \right| = \left| \frac{b_n - c}{cb_n} \right| < \frac{\varepsilon}{|c|r} \tag{2}$$
   for every $n > max\{M, N\}$ □

5. *Define lim inf and lim sup of bounded sequences. Show that a bounded sequence $\{x_n\}_{n\geq 1}$ is convergent if and only if*
   $$\liminf_{n\to\infty}\{x_n\} = \limsup_{n\to\infty}\{x_n\}$$

   Solution: Let $\{x_n\}$ be a bounded sequence. Let $y_n = \inf_{k\geq n}\{x_k\}$ and let $z_n = \sup_{k\geq n}\{x_k\}$, then y_n and z_n are monotone bounded sequences. So they converge. Then $limsup$ and $liminf$ are defined as
   $$limsup\{x_n\} = lim\{z_n\}$$
   $$liminf\{x_n\} = lim\{y_n\}$$
   Let x_n be a convergent sequence and it converges to x,
   let $\epsilon > 0$ be arbitary. Since x_n converges to x, there exists $N \in \mathbb{N}$ such that $x + \epsilon \geq x_n \geq x - \epsilon$ for all $n \geq N$.
   Then, $x + \epsilon \geq y_k \geq x - \epsilon$ for all $k \geq N$.
   Therefore , $x + \epsilon \geq \liminf x_n \geq x - \epsilon$,
   Therefore $\mid \liminf\{x_n\} - x \mid \leq \epsilon$, since ϵ is arbitrary $\liminf x_n = x$.
   Similarly by replacing z_n for y_n, we get $\limsup x_n = x$.

   Now conversely suppose that $\liminf_{n\longrightarrow\infty}\{x_n\} = \limsup_{n\longrightarrow\infty}\{x_n\}$.
   $\inf_{n\geq k}\{x_n\} \leq x_k \leq \sup_{n\geq k}\{x_n\}$. Now proof follows by taking the limit on both inequalities. □

6. *Show that a series of real numbers $\Sigma_{n=1}^{\infty} a_n$ is convergent if $\Sigma_{n=1}^{\infty} \mid a_n \mid$ is convergent and converse is not true.*

   Solution: Let $\epsilon > 0$ be arbitrary and since $\Sigma_{n=1}^{\infty} \mid a_n \mid$ converges, there exists $N \in \mathbb{N}$ such that $\Sigma_{i=n}^{m} \mid a_i \mid \leq \epsilon$ for all $m, n \geq N$.
   Since $\mid \Sigma_{i=n}^{m} a_i \mid \leq \Sigma_{i=n}^{m} \mid a_i \mid \leq \epsilon$ for all $m, n \geq N$.
   Therefore $\Sigma_{n=1}^{\infty} a_n$ converges. To prove the converse need not be true take $a_n = \frac{(-1)^n}{n}$, then $\Sigma_{n=1}^{\infty} \mid a_n \mid$ diverges by Leibeniez test and by p-test $\Sigma_{n=1}^{\infty} a_n$ converges. □

7. *Let z be be the real number with binary expression:*
   $$z = 0.b_1 b_2 b_3 \cdots , \tag{3}$$
   where $b_k = 1$ if $k = 3n + 1$ for some natural number n, and $b_k = 0$ otherwise. Compute the decimal expansion of z.

   Solution: We know that
   $$z = \Sigma_{n=1}^{\infty} \frac{1}{2^{3n+1}} = 1/14 = \Sigma_{n=1}^{\infty} \frac{x_n}{10^n} \tag{4}$$

$$x_n = \begin{cases} 0 & if\ n = 1 \\ 7 & if\ n = 6k - 4 for k \in \mathbb{N} \\ 1 & if\ n = 6k - 3 for k \in \mathbb{N} \\ 4 & if\ n = 6k - 2 for k \in \mathbb{N} \\ 2 & if\ n = 6k - 1 for k \in \mathbb{N} \\ 8 & if\ n = 6k for k \in \mathbb{N} \\ 5 & if\ n = 6k + 1 for k \in \mathbb{N} \end{cases}$$

This can be proved by observing that $1/14 = 0.0\overline{714285}$. □